Lecture 4: Covering and fibration II

G-principal covering

Definition

Let G be a discrete group. A continuous action $G \times X \to X$ is called properly discontinuous if for any $x \in X$, there exists an open neighborhood U of x such that

$$g(U) \cap U = \emptyset, \quad \forall g \neq 1 \in G.$$

We define the orbit space

$$X/G = X/\sim$$

where $x \sim g(x)$ for any $x \in X, g \in G$.

Proposition

Assume G acts properly discontinuously on X, then the quotient map $X \to X/G$ is a covering with fiber G.

Definition

A left (right) G-principal covering is a covering $p : E \to B$ with a left (right) properly discontinuous G-action on E over B

$$\begin{array}{ccc} E & \xrightarrow{\quad g \quad} & E \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle p} & \\ & B & \end{array}, \qquad \forall g \in G$$

such that the induced map $E/G \to B$ is a homeomorphism.

Example

$\exp\colon \mathbb{R}^1 \to S^1$ is a $\mathbb{Z}$-principal covering for the action $n : t \to t + n, \forall n \in \mathbb{Z}$.

Example

$S^n \to \mathbb{RP}^n \simeq S^n/\mathbb{Z}_2$ is a $\mathbb{Z}_2$-principal covering.

Proposition

Let $p : E \to B$ be a G-principal covering. Then transport commutes with G-action, i.e.,

$$T_{[\gamma]} \circ g = g \circ T_{[\gamma]}, \quad \forall g \in G, \gamma \text{ a path in } B.$$

Theorem

Let $p : E \to B$ be a G-principal covering, E path connected, $e \in E, b = p(e)$. Then we have an exact sequence of groups

$$1 \to \pi_1(E, e) \to \pi_1(B, b) \to G \to 1.$$

In other words, $\pi_1(E, e)$ is a normal subgroup of $\pi_1(B, b)$ and

$$G = \pi_1(B, b)/\pi_1(E, e).$$

This can be illustrated by

$$\begin{array}{ccc} & \mathrm{Stab}_e(\pi_1(B,b)\times G) & \\ & {}^{\mathrm{pr}_1}\swarrow \qquad \searrow^{\mathrm{pr}_2} & \\ \pi_1(B,b) & & G \end{array}$$

pr_1 is an isomorphism and pr_2 is an epimorphism with

$$\ker(\mathrm{pr}_2) = \mathrm{Stab}_e(\pi_1(B,b)) = \pi_1(E,e).$$

Example

Apply this Corollary to the covering $\exp\colon \mathbb{R}^1 \to S^1$, we find a group isomorphism (degree map)

$$\deg : \pi_1(S^1) \to \mathbb{Z}.$$

Example

As we will see, S^n is simply connected if $n > 1$. It follows that

$$\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2, \quad n > 1.$$

Applications

Definition

$i : A \subset X$ be a subspace. A continuous map $r : X \to A$ is called a retraction if $r \circ i = 1_A$. It is called a deformation retraction if furthermore we have a homotopy $i \circ r \simeq 1_X$. We say A is a (deformation) retract of X if such a (deformation) retraction exists.

Proposition

If $i : A \subset X$ is a retract, then $r_* : \pi_1(A) \to \pi_1(X)$ is injective.

Corollary

Let D^2 be the unit disk in $\mathbb{R}^2$. Then its boundary S^1 is not a retract of D^2.

Theorem (Brouwer fixed point Theorem)

Let $f : D^2 \to D^2$. Then there exists $x \in D^2$ such that $f(x) = x$.

Theorem (Fundamental Theorem of Algebra)

Let $f(x) = x^n + c_1 x^{n-1} + \cdots + c_n$ be a polynomial with $c_i \in \mathbb{C}, n > 0$. Then there exists $a \in \mathbb{C}$ such that $f(a) = 0$.

Proposition (Antipode)

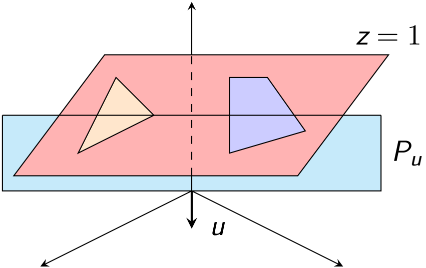
Let $f\colon S^1 \to S^1$ be an antipode-preserving map, i.e. $f(-x) = f(-x)$. Then $\deg(f)$ is odd. In particular, f is NOT null homotopic.

Theorem (Borsuk-Ulam)

Let $f\colon S^2 \to \mathbb{R}^2$. Then there exists $x \in S^2$ such that $f(x) = f(-x)$.

Corollary (Ham Sandwich Theorem)

Let A_1, A_2 be two bounded regions of positive areas in $\mathbb{R}^2$. Then there exists a line which cuts each A_i into half of equal areas.



Classification of coverings

Definition

The universal cover of B is a covering map $p: E \to B$ with E simply connected.

The universal cover is unique (if exists) up to homeomorphism. This follows from the lifting criterion and the unique lifting property of covering maps.

Definition

A space is semi-locally simply connected if for any $x_0 \in X$, there is a neighbourhood U_0 such that the image of the map $i_*\colon \pi_1(U_0, x_0) \to \pi_1(X, x_0)$ is trivial.

We recall the following theorem from point-set topology.

Theorem (Existence of the universal cover)

Assume B is path connected and locally path connected. Then universal cover of B exists if and only if B is semi-locally simply connected space.

Definition

We define the category $\mathrm{Cov}(B)$ of coverings of B where

- an object is a covering map $p: E \to B$
- a morphism between two coverings $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ is a map $f: E_1 \to E_2$ such that the following diagram is commutative

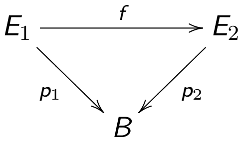
$$\begin{array}{ccc} E_1 & \xrightarrow{\quad f \quad} & E_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & B & \end{array}$$

Definition

Let B be connected. We define $\mathrm{Cov}_0(B) \subset \mathrm{Cov}(B)$ to be the subcategory whose objects consist of connected coverings of B.

Proposition

Let B be connected and locally path connected. Then any morphism in $\mathrm{Cov}_0(B)$ is a covering map.



In other words, if B is connected and p_1, p_2 are coverings, then f is also a covering.

Definition

We define the category G-**Set** where

- an object is a set S with G-action
- morphisms are G-equivariant set maps, i.e. $f\colon S_1 \to S_2$ such that $f \circ g = g \circ f$, for any $g \in G$.

Given a covering $p : E \to B$, $b \in B$, the transport functor implies

$$p^{-1}(b) \in \pi_1(B, b)\text{-}\mathbf{Set}.$$

Lemma

Let B be path connected. Then $\pi_1(B, b)$ acts transitively on $p^{-1}(b)$ if and only if E is path connected.

Proposition

Assume B is path connected and locally path connected. Let $p_1, p_2 \in \mathsf{Cov}(B)$. Then there is a set isomorphism

$$\mathrm{Hom}_{\mathsf{Cov}(B)}(p_1, p_2) \simeq \mathrm{Hom}_{\pi_1(B,b)\text{-}\underline{\mathbf{Set}}}(p_1^{-1}(b), p_2^{-1}(b))$$

for any $b \in B$.

Theorem

Assume B is path connected, locally path connected and semi-locally simply connected. $b \in B$. Then there exists an equivalence of categories

$$\boxed{\mathrm{Cov}(B) \simeq \pi_1(B, b)\text{-}\mathbf{Set}.}$$

The equivalence is realized by the following functors

$$\mathrm{Cov}(B) \underset{G}{\overset{F}{\rightleftarrows}} \pi_1\text{-}\mathbf{Set}.$$

- Let $p : E \to B$ be a covering, we define

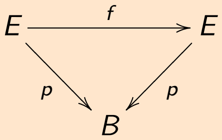
$$F(p) := p^{-1}(b).$$

- Let $S \in \pi_1\text{-}\mathbf{Set}$, we define

$$G(S) := \widetilde{B} \times_{\pi_1} S = \widetilde{B} \times S / \sim .$$

Definition

Let B be path connected and $p : E \to B$ be a connected covering. A deck transformation (or covering transformation) of p is a homeomorphism $f : E \to E$ such that $p \circ f = p$.



Let $\mathrm{Aut}(p)$ denote the group of deck transformation.

Note that $\mathrm{Aut}(p)$ acts freely on E by the Uniqueness of Lifting.

Proposition

Let B be path connected and $p : E \to B$ be a connected covering. Then $\mathrm{Aut}(p)$ acts properly discontinuous on E.

Theorem

Assume B is path connected, locally path connected. Let $p : E \to B$ be a connected covering, $e \in E, b = p(e) \in B$

$$G = \pi_1(B, b), H = \pi_1(E, e).$$

Then

$$\mathrm{Aut}(p) \simeq N_G(H)/H$$

where

$$N_G(H) := \{r \in G \mid rHr^{-1} = H\}$$

is the normalizer of H in G.

This theorem is a direct consequence of the following computation

$$\mathrm{Aut}(p) \simeq \mathrm{Hom}_{G\text{-}\underline{\mathbf{Set}}}(G/H, G/H) = N_G(H)/H.$$

Example

For the universal cover $p : \tilde{B} \to B$, this implies that

$$\mathrm{Aut}(p) = \pi_1(B, b).$$

Therefore p is a $\pi_1(B, b)$-principal covering.

Definition

We define the orbit category $\mathrm{Orb}(G)$

- objects consist of (left) coset G/H, where H is a subgroup of G
- morphisms are G-equivariant maps: $G/H_1 \to G/H_2$.

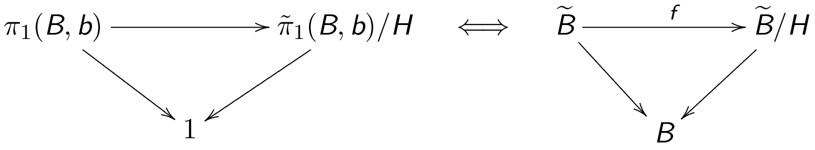
$\mathrm{Orb}(G)$ is a full subcategory of G-**Set** consisting of single orbits.

Remark

G/H_1 and G/H_2 are isomorphic in $\mathrm{Orb}(G)$ if and only if H_1 and H_2 are conjugate subgroups of G.

If we restrict to connected coverings, we find an equivalence

$$\mathrm{Cov}_0(B) \simeq \mathrm{Orb}(\pi_1(B, b)).$$



The universal cover $\widetilde{B} \to B$ corresponds to the orbit $\pi_1(B, b)$. For the orbit $\pi_1(B, b)/H$, it corresponds to

$$E = \widetilde{B}/H \to B.$$

A more intrinsic formulation is as follows. Given a covering $p : E \to B$, we obtain a transport functor

$$T_p : \Pi_1(B) \to \mathbf{Set}.$$

Given a commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & B & \end{array}$$

we find a natural transformation

$$\tau : T_{p_1} \Longrightarrow T_{p_2}, \quad \tau = \{f : p_1^{-1}(b) \to p_2^{-1}(b) | b \in B\}.$$

The above structure can be summarized by a functor

$$\boxed{T : \mathrm{Cov}(B) \to \mathrm{Fun}(\Pi_1(B), \mathbf{Set})}.$$

Theorem

Assume B is path connected, locally path connected and semi-locally simply connected. Then

$$T : \mathrm{Cov}(B) \to \mathrm{Fun}(\Pi_1(B), \mathbf{Set})$$

is an equivalence of categories.